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The Darboux–Bäcklund transformation for the static 2-dimensional continuum Heisenberg chain

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Abstract
We construct the Darboux–Bäcklund transformation for the sigma model describing static configurations of the 2-dimensional classical continuum Heisenberg chain. The transformation is characterized by a non-trivial normalization matrix depending on the background solution. In order to obtain the transformation we use a new, more general, spectral problem.

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1. Introduction

In this paper we consider the 2-dimensional Euclidean $O(3)$ σ-model

$$n_{,xx} + n_{,yy} + \left(n_{,x}^2 + n_{,y}^2\right)n = 0, \qquad n^2 = 1, \tag{1}$$

where $n \in \mathbb{E}^3$ and the comma means differentiation ($n_{,x} = \partial n/\partial x$ etc). This equation appears in classical field theory and solid state physics. Considering the (2+1)-dimensional S^2 σ-model [13]

$$\partial^\mu \partial_\mu \phi + (\partial^\mu \phi \cdot \partial_\mu \phi)\phi, \qquad \phi \cdot \phi = 0, \tag{2}$$

and the (2+1)-dimensional continuum classical Heisenberg ferromagnet equation

$$\vec{S}_{,t} = \vec{S} \times (\vec{S}_{,xx} + \vec{S}_{,yy}), \qquad \vec{S}^2 = 1, \tag{3}$$

we see that their static solutions satisfy (1).

This σ-model plays an important role also in differential geometry. The normal vector n to surfaces of a constant mean curvature endowed with conformal coordinates satisfies (1) [3, 9].

There are many interesting papers on the interpretation of the sigma model (1) and on the construction of special solutions (see, for instance, [1, 3–5, 10, 12, 14]). The most important for physical applications are instanton solutions characterized by the finite total energy

$$\frac{1}{2}\int_{\mathbb{R}^2}\left(n_{,x}^2+n_{,y}^2\right)\mathrm{d}x\,\mathrm{d}y<\infty. \tag{4}$$

They can be characterized as harmonic maps $S^2 \to S^2$ expressed in terms of analytic (rational) functions [1, 4].

The solutions of (1) with infinite total energy are also of some physical interest [11, 12]. The simplest solution of this kind

$$n=\frac{1}{\sqrt{x^2+y^2}}\begin{pmatrix}x\\y\\0\end{pmatrix} \tag{5}$$

is called a meron and is singular at zero and at infinity. Its total energy integral is logarithmically divergent at these points. Merons can be interpreted as point charges and instantons as dipoles made up from meron pairs [11].

Equation (1) can be written in several equivalent forms, for instance $(1-nn^t)(n_{,xx}+n_{,yy})=0$. In this paper we identify the vector n with an $su(2)$ matrix $\mathbf{n}$ (see remark 2). Thus instead of (1) we use an equivalent equation

$$\mathbf{n}_{,xx}+\mathbf{n}_{,yy}=f(x,y)\mathbf{n},\qquad \langle\mathbf{n}|\mathbf{n}\rangle=1, \tag{6}$$

where $\mathbf{n}\in su(2)$, f is a real function and $\langle\mathbf{a}|\mathbf{b}\rangle := -2\,\mathrm{Tr}(\mathbf{ab})$ for $\mathbf{a},\mathbf{b}\in su(2)$. The constraint $\langle\mathbf{n}|\mathbf{n}\rangle=1$ implies that $f(x,y)$ has to be proportional to the energy density, namely $f=-\langle\mathbf{n}_{,x}\mathbf{n}_{,x}\rangle-\langle\mathbf{n}_{,y}\mathbf{n}_{,y}\rangle$.

In this paper we focus on the construction of solutions using the Darboux–Bäcklund transformation. In the case of evolution equations transformations of this kind produce soliton solutions. In this paper we present a large family of gauge-equivalent spectral problems associated with the σ-model (1) and construct the Darboux–Bäcklund transformation for this general spectral problem. The background solution associated with the simplest (constant) solution of the spectral problem is closely related to the meron solution. Therefore special solutions considered in this paper do not satisfy the finite energy condition (4).

2. The spectral problem

We consider the spectral problem of the form

$$\Psi_{,x}=U\Psi\equiv\left(A\zeta-\frac{A^\dagger}{\zeta}+R\right)\Psi,\qquad \Psi_{,y}=V\Psi\equiv\left(B\zeta-\frac{B^\dagger}{\zeta}+S\right)\Psi, \tag{7}$$

(where U, V, Ψ are 2×2 matrices) uniquely characterized by the following properties.

(A) U, V are rational in ζ with simple poles at $\zeta=0$ and $\zeta=\infty$.
(B) $(U(1/\zeta))^\dagger=-U(\bar\zeta)$, $\quad(V(1/\zeta))^\dagger=-V(\bar\zeta)$.
(C) $A^2=B^2=0$.
(D) $B=\mathrm{i}A$.

The constraint (B) implies $R^\dagger=-R$, $S^\dagger=-S$, i.e., R, S are $u(2)$-valued. The compatibility conditions (the coefficient by λ^2) imply that A and B are parallel, i.e.,

$$A=aW,\qquad B=bW, \tag{8}$$

where $a, b \in \mathbb{C}$. Without loss of the generality we can assume $a \in \mathbb{R}$ and

$$a > 0, \qquad \langle W|W^\dagger\rangle = -2, \tag{9}$$

where the scalar product on the space of 2×2 matrices is defined by $\langle X|Y\rangle = -2\,\mathrm{Tr}(XY)$. The coefficient 2 assures that the basis $\mathbf{e}_k \equiv -\mathrm{i}\sigma_k/2$ is orthonormal. We use the standard representation of Pauli matrices, i.e.,

$$\mathbf{e}_1 = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \frac{1}{2}\begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}. \tag{10}$$

The assumptions (9) make the choice of W in equations (8) unique. Indeed, $\langle A|A^\dagger\rangle = |a|^2\langle W|W^\dagger\rangle = -2|a|^2$. Therefore $|a|^2 = -\langle A|A^\dagger\rangle/2 = \mathrm{Tr}(AA^\dagger)$ and, finally,

$$a = \sqrt{\mathrm{Tr}(AA^\dagger)}, \qquad W = A/a. \tag{11}$$

The constraint (D) reduces to $b = \mathrm{i}a$ and is necessary to obtain the standard form of the Laplace operator (geometrically it means that we choose conformal coordinates on the corresponding constant mean curvature surface).

It is convenient to define the following frame:

$$E_1 = \frac{W + W^\dagger}{2\mathrm{i}}, \qquad E_2 = \frac{W^\dagger - W}{2}, \qquad E_3 = [E_1, E_2]. \tag{12}$$

Note that $\langle E_k|E_j\rangle = \delta_{kj}$, $E_k^\dagger = -E_k$ and $\mathrm{Tr}\,E_k = 0$ for $k, j = 1, 2, 3$. Thus this is an orthonormal basis in $su(2)$. Any orthonormal basis in $su(2)$ can be parameterized by a wector W, satisfying $W^2 = 0$ and $\mathrm{Tr}(WW^\dagger) = 1$, according to formulae (12).

The matrices E_1, E_2, E_3 form an orthonormal moving frame in the space $\mathbb{E}^3$ spanned by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2$. Derivatives of E_1, E_2, E_3 are also elements of this space and, as a consequence, they can be expressed as linear combinations of E_1, E_2, E_3. Taking into account that the basis E_1, E_2, E_3 is orthonormal we can reduce the number of coefficients to six real functions $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$:

$$\begin{aligned} \frac{\partial}{\partial x}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} &= \begin{pmatrix} 0 & \alpha_1 & \beta_1 \\ -\alpha_1 & 0 & \gamma_1 \\ -\beta_1 & -\gamma_1 & 0 \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \\ \frac{\partial}{\partial y}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} &= \begin{pmatrix} 0 & \alpha_2 & \beta_2 \\ -\alpha_2 & 0 & \gamma_2 \\ -\beta_2 & -\gamma_2 & 0 \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}. \end{aligned} \tag{13}$$

In other words, identifying the frame E_1, E_2, E_2 with an $SO(3)$-valued function, we immediately see that the kinematics of this frame is expressed by a pair of $so(3)$ matrices. The compatibility conditions for the system (13) read

$$\begin{aligned} &\alpha_{1,y} - \alpha_{2,x} + \beta_2\gamma_1 - \beta_1\gamma_2 = 0, \qquad \beta_{1,y} - \beta_{2,x} + \alpha_1\gamma_2 - \alpha_2\gamma_1 = 0, \\ &\gamma_{1,y} - \gamma_{2,x} + \alpha_2\beta_1 - \alpha_1\beta_2 = 0. \end{aligned} \tag{14}$$

Denoting $\zeta = \exp(-\mathrm{i}\kappa)$ and expressing U, V in terms of E_k, we rewrite the spectral problem (7) as follows:

$$U = aE_2\cos\kappa - aE_1\sin\kappa + R, \qquad V = aE_2\sin\kappa + aE_1\cos\kappa + S, \tag{15}$$

where $u(2)$-valued functions R and S are linear combinations of $\mathrm{i}I$, E_1, E_2, E_3 (with real coefficients), i.e.,

$$S = \mathrm{i}s_0 + s_1E_1 + s_2E_2 + s_3E_3, \qquad R = \mathrm{i}r_0 + r_1E_1 + r_2E_2 + r_3E_3.$$

Remark 1. If κ is real (i.e., $\zeta\bar{\zeta} = 1$), then U, V are $u(2)$-valued, and, as a consequence, Ψ assumes values in the group $U(2)$.

The compatibility conditions for the spectral problem (15) read

$$\begin{aligned}
&(aE_2)_{,y} - (aE_1)_{,x} + a[E_2, S] - a[E_1, R] = 0,\\
&(aE_2)_{,x} + (aE_1)_{,y} + a[E_1, S] + a[E_2, R] = 0,\\
&R_{,y} - S_{,x} + [R, S] = a^2E_3.
\end{aligned} \tag{16}$$

The system (16) can be rewritten as follows:

$$\begin{aligned}
&r_1 - \gamma_1 = \beta_2 + s_2, \qquad r_2 + \beta_1 = \gamma_2 - s_1,\\
&a_{,x} + a\alpha_2 - as_3 = 0, \qquad a_{,y} - a\alpha_1 + ar_3 = 0, \qquad r_{0,y} = s_{0,x},\\
&r_{1,y} - s_{1,x} + r_2s_3 - r_3s_2 - \alpha_2r_2 - \beta_2r_3 + \alpha_1s_2 + \beta_1s_3 = 0,\\
&r_{2,y} - s_{2,x} + r_3s_1 - r_1s_3 + \alpha_2r_1 - \gamma_2r_3 - \alpha_1s_1 + \gamma_1s_3 = 0,\\
&r_{3,y} - s_{3,x} + r_1s_2 - r_2s_1 + \beta_2r_1 + \gamma_2r_2 - \beta_1s_1 - \gamma_1s_2 = 4a^2.
\end{aligned} \tag{17}$$

Proposition 1. *Let $\zeta\bar{\zeta} = 1$ and Ψ satisfy (7) and (A)–(D), and let E_k be defined by (12). Then Ψ takes the values in $U(2)$ and*

$$\mathbf{n} = \Psi^{-1}E_3\Psi \tag{18}$$

satisfies equation (6).

Remark 2. We use the isomorphism between $su(2)$ and $\mathbb{E}^3$. The coefficients of the matrix $\mathbf{n} \in su(2)$ with respect to the basis (10), $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$, identify the matrix $\mathbf{n}$ with the vector $n = (n_1, n_2, n_3) \in \mathbb{E}^3$ which solves (1).

Proof. The derivatives of $\mathbf{n}$ can be computed as

$$\mathbf{n}_{,x} = \Psi^{-1}(E_{3,x} + [E_3, U])\Psi, \qquad \mathbf{n}_{,y} = \Psi^{-1}(E_{3,y} + [E_3, V])\Psi,$$

which yields

$$\begin{aligned}
\mathbf{n}_{,x} &= \Psi^{-1}(-\beta_1E_1 - \gamma_1E_2 - aE_1\cos\kappa - aE_2\sin\kappa + r_1E_2 - r_2E_1)\Psi,\\
\mathbf{n}_{,y} &= \Psi^{-1}(-\beta_2E_1 - \gamma_2E_2 + aE_2\cos\kappa - aE_1\sin\kappa + s_1E_2 - s_2E_1)\Psi.
\end{aligned}$$

In general, if $\Psi = \Psi(x, y)$ is $U(2)$-valued and $E = E(x, y)$ takes values in $su(2)$, then $\Psi^{-1}E\Psi$ takes values in $su(2)$ as well. Therefore, computing $\mathbf{n}_{,xx} + \mathbf{n}_{,yy}$ we obtain a linear combination of $\Psi^{-1}E_k\Psi$ $(k = 1, 2, 3)$. It is enough to show that the result is proportional to $\Psi^{-1}E_3\Psi$, i.e., that the coefficients by $\Psi^{-1}E_1\Psi$ and $\Psi^{-1}E_2\Psi$ vanish. The coefficient by $\Psi^{-1}E_2\Psi$ is given by

$$\begin{aligned}
&(r_1 - \gamma_1)_{,x} - (\gamma_2 - s_1)_{,y} + (\beta_1 + r_2)(r_3 - \alpha_1) + (\beta_2 + s_2)(s_3 - \alpha_2)\\
&\qquad + (a_{,y} - a\alpha_1 + ar_3)\cos\kappa + (a_{,x} + a\alpha_2 - as_3)\sin\kappa.
\end{aligned}$$

To show that this expression vanishes we use the first two equations of the system (17), then we eliminate all derivatives using appropriate equations of (14) and (17). Using once more (if necessary) the first equation of (17) we see that the obtained result is zero. The coefficient by $\Psi^{-1}E_1\Psi$

$$-(r_2+\beta_1)_{,x} - (\beta_2+s_2)_{,y} + (s_1-\gamma_2)(s_3-\alpha_2) + (r_3-\alpha_1)(r_1-\gamma_1) \\ - (a_{,x} + a\alpha_2 - as_3)\cos\kappa - (a_{,y} + ar_3 - a\alpha_1)\sin\kappa$$

vanishes as well, which can be shown in exactly the same way. □

Remark 3. If one more constraint, namely $\operatorname{Tr} U = \operatorname{Tr} V = 0$, $\det\Psi = 1$, is imposed on the linear problem (7), then the Sym–Tafel formula $F = \Psi^{-1}\Psi_{,\kappa}$ yields surfaces of a constant mean curvature (compare [7, 10]).

3. Gauge transformations

The spectral problem (7) is invariant with respect to gauge transformations of the form $\hat\Psi = G\Psi$, where G is any ζ-independent $U(2)$-valued matrix ($G^{-1} = G^\dagger$).

Proposition 2. *If Ψ satisfies (7), (A)–(D) and $\hat\Psi = G\Psi$, where $G^{-1} = G^\dagger$, then $\hat\Psi$ satisfies (7), (A)–(D) as well. Moreover*

$$\hat{\mathbf{n}} \equiv \hat\Psi^{-1}\hat E_3\hat\Psi = \mathbf{n}. \tag{19}$$

Proof. $\hat\Psi_{,x} = \hat U\hat\Psi$, $\hat\Psi_{,y} = \hat V\hat\Psi$, where

$$\hat U = GAG^{-1}\zeta - \frac{GA^\dagger G^{-1}}{\zeta} + R + G_{,x}G^{-1} \equiv \hat A\zeta - \frac{\hat A^\dagger}{\zeta} + \hat R,$$
$$\hat V = GBG^{-1}\zeta - \frac{GB^\dagger G^{-1}}{\zeta} + S + G_{,y}G^{-1} \equiv \hat B\zeta - \frac{\hat B^\dagger}{\zeta} + \hat S,$$

where $(GAG^{-1})^\dagger = GA^\dagger G^{-1}$ because $G^\dagger = G^{-1}$. Obviously, $\hat A^2 = \hat B^2 = 0$, $\hat R^\dagger = -\hat R$, $\hat S^\dagger = -\hat S$, etc. Thus the matrices $\hat U$, $\hat V$ satisfy all conditions (A)–(D). Then $\operatorname{Tr}(\hat A\hat A^\dagger) = \operatorname{Tr}(GAG^{-1}GA^\dagger G^{-1}) = \operatorname{Tr}(AA^\dagger)$. Hence, taking into account (11) and (12), we get $\hat a = a$, $\hat W = GWG^{-1}$ and $\hat E_k = GE_kG^{-1}$ $(k = 1, 2, 3)$. Finally, $\hat{\mathbf{n}} = \Psi^{-1}G^{-1}GE_3G^{-1}G\Psi = \mathbf{n}$. □

Proposition 3. *There exists a matrix $G = G(x, y) \in U(2)$ transforming the spectral problem (7), (A)–(D) into*

$$\hat\Psi_{,x} = \left(a\mathbf{e}_+\zeta - \frac{a\mathbf{e}_-}{\zeta} + \hat R\right)\hat\Psi, \\ \hat\Psi_{,y} = \left(\mathrm{i}a\mathbf{e}_+\zeta + \frac{\mathrm{i}a\mathbf{e}_-}{\zeta} + \hat S\right)\hat\Psi, \tag{20}$$

where a is given by (11) and

$$\mathbf{e}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{e}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{21}$$

Proof. Any two orthonormal bases in $\mathbb{E}^3$ are related by an orthogonal transformation, which in turn can be represented by a unitary matrix (the spinor representation). In particular, the basis E_1, E_2, E_3 from section 2 can be obtained from any constant orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ by the transformation of the form $E_k = G^{-1}\mathbf{e}_kG$, $G \in U(2)$ (or even $G \in SU(2)$, if both bases have the same orientation). Applying the gauge transformation $\hat\Psi = G\Psi$

to the spectral problem (7) we obtain (20), where $\mathbf{e}_+, \mathbf{e}_-$ are constant matrices such that $\mathbf{e}_- = \mathbf{e}_+^\dagger$, $\mathbf{e}_+^2 = \mathbf{e}_-^2 = 0$, $\langle \mathbf{e}_+ | \mathbf{e}_- \rangle = -2$. If $\mathbf{e}_k$ are given by (10), then $\mathbf{e}_\pm$ are given by (21). □

Remark 4. If $\hat{\Psi}$ solves (20), then $\mathbf{n} = \hat{\Psi}^{-1}\mathbf{e}_3\hat{\Psi}$ satisfies (6).

Remark 5. The spectral problem (20) or its equivalents are usually applied in the spectral approach to constant mean curvature surfaces; see [3, 7, 10].

4. The Darboux–Bäcklund transformation

Our aim is to construct the transformation $\tilde{\Psi} = D\Psi$ (where D depends on x, y and ζ) in such a way that $\tilde{U} = D_{,x}D^{-1} + DUD^{-1}$ and $\tilde{V} = D_{,y}D^{-1} + DVD^{-1}$ have the same form as U, V (compare [6]). In other words, properties (A), (B), (C) and (D) of section 2 should be preserved by the transformation. We confine ourselves to the simplest case

$$D = \mathcal{N}\left(I + \frac{\zeta_1 - \mu_1}{\zeta - \zeta_1}P\right), \tag{22}$$

where the matrices $\mathcal{N}$ and P do not depend on ζ, $P^2 = P$, and ζ_1, μ_1 are complex parameters $(\zeta_1 \neq \mu_1)$.

Property (A) implies, by virtue of a well-known result of Zakharov and Shabat [15],

$$\ker P \ni \Psi(\zeta_1)\vec{b}, \qquad \operatorname{Im} P \ni \Psi(\mu_1)\vec{c}, \tag{23}$$

where $\vec{b}, \vec{c} \in \mathbb{C}^2$ are constant vectors and $\zeta_1, \mu_1 \in \mathbb{C}$ are constant as well.

One can easily check that property (B) is preserved if $D^{-1}(\bar{\zeta}) = D^\dagger(1/\zeta)$ which yields, after straightforward computations,

$$P^\dagger = P, \qquad \bar{\mu}_1 = \frac{1}{\zeta_1}, \qquad \mathcal{N}\mathcal{N}^\dagger = 1 + (|\zeta_1|^2 - 1)P. \tag{24}$$

Therefore the condition $\mu_1 \neq \zeta_1$ is equivalent to $|\zeta_1| \neq 1$. Moreover, $P^\dagger = P$ implies $\vec{c}_1 \perp \vec{b}_1$. P is explicitly expressed by the matrix $\Psi(1/\bar{\zeta}_1)$:

$$P = \frac{1}{1 + |\xi|^2}\begin{pmatrix} |\xi|^2 & \xi \\ \bar{\xi} & 1 \end{pmatrix}, \tag{25}$$

where $\xi = u_1/u_2$ and $(u_1, u_2)^T = \Psi(1/\bar{\zeta}_1)\vec{c}$. One can check that the equation $\mathcal{N}\mathcal{N}^\dagger = 1 + (|\zeta_1|^2 - 1)P$ is satisfied by

$$\mathcal{N} = \mathcal{N}_0(I + (\zeta_1\,\mathrm{e}^{\mathrm{i}\sigma} - 1)P), \tag{26}$$

where $\mathcal{N}_0$ is a unitary matrix $\left(\mathcal{N}_0^{-1} = \mathcal{N}_0^\dagger\right)$ and σ is a real constant.

Considering the spectral problem (20) we have to take into account one more constraint: $W = \mathbf{e}_+$ is a fixed constant matrix, given for instance by (21). In this case

$$\tilde{a}\mathbf{e}_+ = a\mathcal{N}\mathbf{e}_+\mathcal{N}^{-1}, \tag{27}$$

and from (27) we can compute $\mathcal{N}_0$.

In the following we focus on the more general spectral problem (7) and the matrix $\mathcal{N}_0$ can be arbitrary. Actually, the matrix $\mathcal{N}_0$ is not important as far as the transformation of $\mathbf{n}$ is concerned (compare proposition 2). Without loss of the generality we will assume $\mathcal{N}_0 = I$. Finally we arrive at following formula for the Darboux matrix:

$$D = (I + (\zeta_1\,\mathrm{e}^{\mathrm{i}\sigma} - 1)P)\left(I + \frac{\zeta_1 - \bar{\zeta}_1^{-1}}{\zeta - \zeta_1}P\right) = I + (\mathrm{e}^{2\mathrm{i}\beta} - 1)P, \tag{28}$$

where

$$e^{2i\beta} := \frac{(\zeta_1 - \zeta|\zeta_1|^2)\, e^{i\sigma_1}}{|\zeta_1|^2 - \zeta\bar{\zeta}_1}. \tag{29}$$

Note that β is real (because $\bar{\zeta} = \zeta^{-1}$) and β does not depend on x, y.

One can always parameterize the Hermitian projector P by a unit vector $\vec{p} = (p_1, p_2, p_3)$:

$$P = \frac{1}{2}(I + \mathbf{p}), \qquad \mathbf{p} := \sum_{k=1}^{3} p_k \sigma_k, \qquad p_1^2 + p_2^2 + p_3^2 = 1. \tag{30}$$

The function ξ appearing in (25) is a stereographic projection of $\vec{p}$:

$$\xi = \frac{p_1 - i p_2}{1 - p_3}, \qquad p_1 = \frac{2\,\mathrm{Re}\,\xi}{1 + |\xi|^2}, \qquad p_2 = \frac{-2\,\mathrm{Im}\,\xi}{1 + |\xi|^2}, \qquad p_3 = \frac{|\xi|^2 - 1}{|\xi|^2 + 1}. \tag{31}$$

The spectral problem (20) can be considered as a particular case of (7) and any solution Ψ of (20) satisfies (7) as well (compare remark 4). Therefore we can take as a background solution

$$\mathbf{n} = \Psi^{-1}\mathbf{e}_3\Psi, \tag{32}$$

where Ψ is a solution of (20). According to remark 2 we associate with $\mathbf{n}$ a unit vector $\vec{n} := (n_1, n_2, n_3)$ defined by

$$\mathbf{n} = \sum_{k=1}^{3} n_k \mathbf{e}_k. \tag{33}$$

The Darboux–Bäcklund transformation of $\mathbf{n}$ yields

$$\tilde{\mathbf{n}} = \Psi^{-1} D^{-1}\mathbf{e}_3 D\Psi. \tag{34}$$

The obtained expression can be computed as follows,

$$D^{-1}\mathbf{e}_3 D = \frac{1}{2i}(\cos\beta - i\mathbf{p}\sin\beta)\sigma_3(\cos\beta + i\mathbf{p}\sin\beta),$$

and simplified in a straightforward way:

$$D^{-1}\mathbf{e}_3 D = \frac{1}{2i}(\sigma_3\cos 2\beta + 2p_3\mathbf{p}\sin^2\beta + (p_2\sigma_1 - p_1\sigma_2)\sin 2\beta). \tag{35}$$

Iterating M times the Darboux-Bäcklund transformation (22), (24) we obtain

$$D_M(\zeta) = \prod_{k=M}^{1} \mathcal{N}_k \left(I + \frac{\zeta_k - \mu_k}{\zeta - \zeta_k} P_k \right) = \mathcal{N}\left(I + \sum_{k=1}^{M} \frac{A_k}{\zeta - \zeta_k} \right), \tag{36}$$

where $\mu_k = \bar{\zeta}_k^{-1}$, $A_k = A_k(x, y)$ are some matrices and $\mathcal{N} = \mathcal{N}_M \cdots \mathcal{N}_2\mathcal{N}_1$ (which can be shown by taking the limit $\zeta \to \infty$). Similarly

$$D_M^{-1}(\zeta) = \prod_{k=1}^{M} \left(I + \frac{\mu_k - \zeta_k}{\zeta - \mu_k} P_k \right) \mathcal{N}_k^{-1} = \left(I + \sum_{k=1}^{M} \frac{B_k}{\zeta - \mu_k} \right) \mathcal{N}^{-1}, \tag{37}$$

where $B_k = B_k(x, y)$ are some matrices. The reduction $D_M^{-1}(\zeta) = D_M^{\dagger}(1/\bar{\zeta})$ yields

$$\mathcal{N}^{\dagger}\mathcal{N} = I - \sum_{k=1}^{M} \frac{B_k}{\mu_k} = \left(I - \sum_{k=1}^{M} \mu_k A_k \right)^{-1}, \qquad B_k = -\mu_k^2 A_k^{\dagger}\mathcal{N}^{\dagger}\mathcal{N}, \tag{38}$$

which generalizes formulae (24).

5. Special solutions

We will compute explicitly the action of the Darboux–Bäcklund transformation on a simple background. The simplest seed solution can be obtained from the requirement $U = \text{const}$, $V = \text{const}$ and Ψ satisfies (20). Then $E_k = \mathbf{e}_k$ are constant (i.e., $\alpha_k = \beta_k = \gamma_k = 0$) and $a = a_0 = \text{const}$. Thus the system (17) reduces to

$$r_1 = s_2, \qquad r_2 = -s_1, \qquad s_3 = r_3 = 0, \qquad r_1 s_2 - r_2 s_1 = a_0^2, \tag{39}$$

and can easily be solved:

$$s_1 = -r_2 = a_0 \cos\delta_0, \qquad r_1 = s_2 = a_0 \sin\delta_0, \tag{40}$$

where δ_0 is an arbitrary real parameter. Therefore,

$$\begin{aligned} U &= a_0 \mathbf{e}_2(\cos\kappa - \cos\delta_0) - a_0 \mathbf{e}_1(\sin\kappa - \sin\delta_0), \\ V &= a_0 \mathbf{e}_1(\cos\kappa + \cos\delta_0) + a_0 \mathbf{e}_2(\sin\kappa + \sin\delta_0), \end{aligned} \tag{41}$$

and, finally

$$U = 2a_0 \sin\delta_-(\mathbf{e}_1 \cos\delta_+ + \mathbf{e}_2 \sin\delta_+), \qquad V = 2a_0 \cos\delta_-(\mathbf{e}_1 \cos\delta_+ + \mathbf{e}_2 \sin\delta_+), \tag{42}$$

where $\delta_\pm := \frac{1}{2}(\delta_0 \pm \kappa)$. Without loss of the generality we put $\delta_0 = 0$ (more general choice corresponds to symmetries of the sigma model (1) like rotation in the space of parameters x, y and the $O(3)$ symmetry). Then

$$-2\mathbf{e}_1 \cos\delta_+ - 2\mathbf{e}_2 \sin\delta_+ = \mathrm{i}\begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\kappa/2} \\ \mathrm{e}^{\mathrm{i}\kappa/2} & 0 \end{pmatrix} =: E. \tag{43}$$

Note that $E^2 = -1$. Therefore, if U, V are constant, then the solution of the linear problem (7) is simply given by

$$\Psi = \exp(xU + yV)C_0 = \exp(\theta E)C_0 = (\cos\theta + E\sin\theta)C_0, \tag{44}$$

where C_0 is a constant unitary matrix and

$$\theta = \theta(x, y, \zeta) = a_0 x \sin\frac{\kappa}{2} - a_0 y \cos\frac{\kappa}{2}. \tag{45}$$

Thus, taking into account $\zeta = \mathrm{e}^{-\mathrm{i}\kappa}$,

$$\Psi(x, y, \zeta) = \begin{pmatrix} \cos\theta & \mathrm{i}\sqrt{\zeta}\sin\theta \\ \mathrm{i}\sin\theta/\sqrt{\zeta} & \cos\theta \end{pmatrix} C_0. \tag{46}$$

Finally, using (32), we get the following background solution:

$$\mathbf{n} = \mathbf{e}_1 \sin 2\theta \sin\frac{\kappa}{2} - \mathbf{e}_2 \sin 2\theta \cos\frac{\kappa}{2} + \mathbf{e}_3 \cos 2\theta. \tag{47}$$

The energy density of this solution is constant

$$\tfrac{1}{2}(\langle \mathbf{n}_{,x} | \mathbf{n}_{,x}\rangle + \langle \mathbf{n}_{,y} | \mathbf{n}_{,y}\rangle) = 2a_0^2, \tag{48}$$

and the topological charge density is zero

$$\frac{1}{4\pi}\langle \mathbf{n} | [\mathbf{n}_{,x}, \mathbf{n}_{,y}]\rangle = 0. \tag{49}$$

The background solution (47) is related to the meron solution (5) by the conformal transformation

$$\tilde{x} + \mathrm{i}\tilde{y} = \exp(-2a_0\,\mathrm{e}^{-\mathrm{i}\kappa/2}(x + \mathrm{i}y)), \tag{50}$$

and a rotation in the space $\mathbb{E}^3$.

Now, we will perform the Darboux–Bäcklund transformation. In order to compute ξ we evaluate Ψ at $\zeta = 1/\bar{\zeta}_1$ and denote $\lambda_1 := 1/\sqrt{\bar{\zeta}_1}$:

$$\Psi\big(x, y, \bar{\zeta}_1^{-1}\big) = \begin{pmatrix} \cos\theta_1 & -\mathrm{i}\lambda_1 \sin\theta_1 \\ -\mathrm{i}\lambda_1^{-1}\sin\theta_1 & \cos\theta_1 \end{pmatrix} C_0, \tag{51}$$

where $\theta_1 = \theta\big(x, y, \zeta_1^{-1}\big) \equiv P_1 + \mathrm{i}Q_1$, i.e.,

$$P_1 = -\frac{1}{2}a_0\left(1 + \frac{1}{a_1^2 + b_1^2}\right)(xb_1 + ya_1), \qquad Q_1 = \frac{1}{2}a_0\left(1 - \frac{1}{a_1^2 + b_1^2}\right)(xa_1 - yb_1), \tag{52}$$

where $a_1 + \mathrm{i}b_1 := \lambda_1 \equiv \zeta_1^{-1/2}$ (we recall that by assumption $a_1^2 + b_1^2 \neq 1$). Then

$$\xi = \frac{c_1 - \mathrm{i}c_2\lambda_1\tan\theta_1}{c_2 - \mathrm{i}c_1\lambda_1^{-1}\tan\theta_1}, \tag{53}$$

where $(c_1, c_2)^T = C_0\vec{c}$. Without loss of the generality we can put $c_1 = 0$ (one can show that more general choice is equivalent to a translation in the space of variables x, y, compare [2]). Then, finally,

$$\xi = \frac{(a_1 + \mathrm{i}b_1)(\sinh Q_1\cosh Q_1 - \mathrm{i}\sin P_1\cos P_1)}{\cosh^2 Q_1\cos^2 P_1 + \sinh^2 Q_1\sin^2 P_1}, \tag{54}$$

where P_1, Q_1 are given by (52) and a_1, b_1 are arbitrary real parameters.

Therefore the solution $\tilde{\mathbf{n}}$ given by (34) can be easily computed using (35), (31), (46), (54) and (52), where $\beta, a_0, a_1, b_1, \kappa$ and the matrix C_0 are arbitrary constants. In particular, assuming $C_0 = I$ and $\kappa = 0$ we obtain $\tilde{n} = (n_1, n_2, n_3)$, where

$$\begin{aligned} n_1 &= 2p_1p_3\sin 2\beta + p_2\sin 2\beta, \\ n_2 &= (2p_2p_3\sin^2\beta - p_1\sin 2\beta)\cos 2\theta - \big(\cos 2\beta + 2p_3^2\sin^2\beta\big)\sin 2\theta, \\ n_3 &= (2p_2p_3\sin^2\beta - p_1\sin 2\beta)\sin 2\theta + \big(\cos 2\beta + 2p_3^2\sin^2\beta\big)\cos 2\theta. \end{aligned} \tag{55}$$

The functions p_1, p_2, p_3 are given by (31) and (54), θ is given by (45).

6. Conclusions

In this paper, we presented a new version of the Darboux–Bäcklund transformation for the sigma model (1). There are two interesting points in our construction. First, we introduced the spectral problem (7), more general than (20). Both spectral problems are gauge-equivalent and the sigma model (1) is invariant with respect to unitary gauge transformations of the spectral problem (compare proposition 2). Second, the normalization matrix (26) is quite non-trivial. The matrix $\mathcal{N}$ depends on x, y through the projector matrix P (i.e., through the background wavefunction). Note that the Darboux–Bäcklund transformation for the spectral problem (20) is even more difficult. We have an additional constraint on the unitary matrix $\mathcal{N}_0$, namely (27), which is technically pretty complicated. From this point of view the spectral problem (7) is more convenient.

Our approach is rather straightforward and we plan to generalize it on some related sigma models and geometric problems (surfaces of constant mean curvature in Euclidean and Lorentzian spaces). We expect to produce some new results considering spectral problems of the form (7) but with matrices of higher dimension. An especially promising way to approach higher dimensional problems consists of replacing matrices by Clifford numbers (compare [8]).

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